SEMESTER – II

UNIT – I

Functions and their Applications

(1) The total cost of a firm is Rs. 20,000 when there is no production and Rs. 80,000 when the output is 500 units. The relationship of total cost to the output is linear. Obtain the formula for the total cost.

Solution:

Let C(x) be the cost of producing x units.

$\therefore$ The cost to the output is linear. Let C(x) = a + bx.

where a and b are constants.

Now, $C(x) = 20,000$ when $x = 0$

and $C(x) = 80,000$ when $x = 1500$

$$\therefore 20,000 = a + b(0) \text{ and } 80000 = a + 1500\,b$$

$$\therefore 20,000 = a$$

Substituting value of a in $80,000 = a + 1500\,b$

$$80,000 = 20,000 + 1500\,b$$

$$80,000 - 20,000 = 1500\,b$$

$$1500\,b = 60,000$$

$$b = \frac{60000}{1500}$$

$$\therefore b = 40$$

We get,

The formula of total cost as,

$$C(x) = 20,000 + 40x$$

where a = 20,000 is the fixed cost and b = 40 is the variable cost.

(2) If y is the quantity and x is the price of a commodity. The demand and supply curves are given by linear equations $2x + y = 600$ and $3x - y = 400$. Find the equilibrium price and the corresponding quantity.

Solution:

$$2x + y = 600 \Rightarrow y = 600 - 2x$$

$$3x - y = 400 \Rightarrow y = 3x - 400$$

At equilibrium, Demand = Supply

$$\Rightarrow \quad 600 - 2x = 3x - 400$$

$$600 + 400 = 3x + 2x$$

$$1{,}000 = 5x$$

$$x = \frac{1000}{5}$$

$$x = 200$$

Put $x = 200$ in $y = 3x - 400$,

$$y = 3(200) - 400$$

$$y = 600 - 400$$

$$y = 200$$

$\therefore$ The equilibrium price is 200 and equilibrium quantity is 200.

(3) One of the following curve represents supply function and the other represent demand function $x = 5 + 3p - p^2$ and $x = p^2 + p + 1$, where p = price and x = quantity. State with reasons, which is the supply function? Also find the equilibrium price and the quantity demanded at that time.

Solution:

At equilibrium, Demand = Supply

$$\Rightarrow \quad 5 + 3p - p^2 = p^2 + p + 1$$

$$5 - 1 + 3p - p = p^2 + p^2$$

$$4 + 2p = 2p^2$$

$$2p^2 - 2p - 4 = 0$$

Divide throughout by 2,

$$p^2 - p - 2 = 0$$

$$p^2 - 2p + p - 2 = 0$$

$$p(p - 2) + 1\,(p - 2) = 0$$

$$(p - 2)\,(p + 1) = 0$$

$$p = 2 \quad \text{or } p = -1$$

But $p \nless 0$.

So $p \neq -1$ is discarded.

Hence $\boxed{p = 2}$

Put $p = 2$ in $\quad x = p^2 + p + 1$

$$x = (2)^2 + 2 + 1$$

$$x = 4 + 2 + 1$$

$$x = 7$$

So, the equilibrium price is 2 and the equilibrium quantity is 7.

(4) A toy manufacturing company has 29 workers. The fixed cost turns out to be Rs. 30 per worker daily. The cost of producing a toy is Rs. 2.07 per unit. If each toy is sold for Rs. 6 then determine the profit function. What is the profit if 1000 units are produced and sold in a day? Also find the number of toys that should be produced and sold to ensure no loss.

Solution:

$$FC = \text{`}30 \times 29 = 870$$

$$VC = \text{`}2.07/\text{unit.}$$

Sales price = `6/unit.

Let x be the number of toys produced and sold.

VC for 1 unit is `2.07.

So, VC for x unit is `2.07x.

Total cost function:

$$C(x) = VC\ (x) + FC$$

$$C(x) = 2.07x + 870$$

Revenue function:

$$\begin{aligned} R(x) &= \text{Sales price of 1 unit} \times \text{No. of units} \\ &= 6 \times x \\ R(x) &= 6x \end{aligned}$$

Profit function:

$$\begin{aligned} \pi(x) &= R(x) - C(x) \\ &= 6x - (2.07x + 870) \\ &= 6x - 2.07x - 870 \\ \pi(x) &= 3.93x - 870 \end{aligned}$$

For $x = 1000$

$$\begin{aligned} \pi(1000) &= 3.93\ (1000) - 870 \\ &= 3930 - 870 \\ \pi(1000) &= \text{`}3060 \end{aligned}$$

For no loss,

$$\pi(x) = 0 \qquad \Rightarrow 3.93x - 870 = 0$$

$$3.93x = 870$$

$$x = \frac{870}{3.93} \quad \Rightarrow x = 221.374$$

$\therefore$ To ensure no loss at least 222 units of toy must be produced and sold.

(5) If the total revenue function and the total cost function for a product are given by $R(x) = 100x - 2x^2$ and $C(x) = 10x + 700$, find the least number of units that should be produced to avoid a loss. Also find the maximum number of products that will still give a gain.

Solution:

$$R(x) = 100x - 2x^2 \qquad C(x) = 10x + 700$$

At break-even point

$$\begin{aligned} R(x) &= C(x) \\ 100x - 2x^2 &= 10x + 700 \\ 100x - 10x &= 2x^2 + 700 \\ 90x &= 2x^2 + 700 \\ 2x^2 - 90x + 700 &= 0 \end{aligned}$$

Dividing throughout by 2,

$$\begin{aligned} x^2 - 45x + 350 &= 0 \\ x^2 - 35x - 10x + 350 &= 0 \\ x(x - 35) - 10(x - 35) &= 0 \\ (x - 35)(x - 10) &= 0 \\ \Rightarrow x = 35 & \text{ or } x = 10 \end{aligned}$$

$\therefore$ At least 10 units must be produced to avoid a loss and at most 34 units of product will give a gain.

Derivatives

Find dy/dx for the following:

(1) $y = 3x^7 - 5x^4 + 9x^2$

Solution:

Diff. w.r.t. x

$$\frac{d(y)}{dx} = \frac{d}{dx}(3x^7 - 5x^4 + 9x^2)$$

$$\frac{dy}{dx} = 3\frac{d}{dx}(x^7) - 5\frac{d}{dx}(x^4) + 9\frac{d}{dx}(x^2)$$

$$\frac{dy}{dx} = 3(7x^6) - 5(4x^3) + 9(2x)$$

$$\frac{dy}{dx} = 21x^6 - 20x^3 + 18x$$

(2) $y = 4x^{1/6} - 3x^{4/6} - x^{1/6}$

Solution:

Diff. w.r.t. x

$$\frac{d(y)}{dx} = \frac{d}{dx}(4x^{1/6} - 3x^{4/6} - x^{1/6})$$

$$\frac{dy}{dx} = 4\frac{d}{dx}(x^{1/6}) - 3\frac{d}{dx}(x^{4/6}) - \frac{d}{dx}(x^{1/6})$$

$$\frac{dy}{dx} = 4\left(\frac{1}{6}\right)x^{-5/6} - 3\left(\frac{4}{6}\right)x^{-2/6} - \left(\frac{1}{6}\right)x^{-5/6}$$

$$\frac{dy}{dx} = \frac{4}{6x^{5/6}} - \frac{12}{6x^{2/6}} - \frac{1}{6x^{5/6}}$$

(3) $y = 8x^4 - \log x^4 + 6e^x + 5^x$

Solution:

$$y = 8x^4 - 4\log x + 6e^x + 5^x$$

Diff. w.r.t. x

$$\frac{d(y)}{dx} = \frac{d}{dx}(8x^4 - 4\log x + 6e^x + 5^x)$$

$$\frac{dy}{dx} = 8\frac{d}{dx}(x^4) - 4\frac{d}{dx}(\log x) + 6\frac{d}{dx}(e^x) + \frac{d}{dx}(5^x)$$

$$\frac{dy}{dx} = 8(4x^3) - 4\left(\frac{1}{x}\right) + 6e^x + 5^x\log 5$$

$$\frac{dy}{dx} = 32x^3 - \frac{4}{x} + 6e^x + 5^x\log 5$$

(4) $y = 5x^8 - 5^x - x^{5/2} + \log a^3$

Solution:

Diff. w.r.t. x

$$\frac{d(y)}{dx} = \frac{d}{dx}(5x^8 - 5x - x^{5/2} + \log a^3)$$

$$\frac{dy}{dx} = 5\frac{d}{dx}(x^8) - \frac{d}{dx}(5^x) - \frac{d}{dx}(x^{5/2}) + \frac{d}{dx}(\log a^3)$$

$$\frac{dy}{dx} = 5(8x^7) - 5^x\log 5 - \left(\frac{5}{2}\right)x^{3/2} + 0$$

$$\frac{dy}{dx} = 40x^7 - 5^x\log 5 - \frac{5x^{3/2}}{2}$$

(5) $y = 7x^{1/4} + \log 5x - a^2$

Solution:

Diff. w.r.t. x

$$\frac{d(y)}{dx} = \frac{d}{dx}(7x^{1/4} + \log 5x - a^2)$$

$$\frac{dy}{dx} = 7\frac{d}{dx}(x^{1/4}) - \frac{d}{dx}(\log 5x) - \frac{d}{dx}(a^2)$$

$$\frac{dy}{dx} = 7\frac{d}{dx}(x^{1/4}) - \frac{d}{dx}[\log 5 + \log x] - \frac{d}{dx}(a^2)$$

$$\frac{dy}{dx} = 7\frac{d}{dx}(x^{1/4}) - \frac{d}{dx}(\log 5) + \frac{d}{dx}(\log x) - \frac{d}{dx}(a^2)$$

(6) $y = (4x^4 - 3x^2 + 9x + 8)(2e^x - 2\log x)$

Solution:

Diff. w.r.t. x

$$y = uv, \frac{dy}{dx} = u\frac{d}{dx}v + v\frac{d}{dx}u$$

$$\frac{dy}{dx} = (4x^4 - 3x^2 + ax + 8)\frac{d}{dx}(2e^x - 2\log x)$$

$$+ (2e^x - 2\log x)\frac{d}{dx}(4x^4 - 3x^2 + 9x + 8)$$

$$\frac{dy}{dx} = (4x^4 - 3x^2 + 9x + 8)\left[2\frac{d}{dx}(e^x) - 2\frac{d}{dx}\right.$$

$$\frac{dy}{dx} = 7\left(\frac{1}{4}\right)x^{-3/4} + 0 + \frac{1}{x} - 0$$

$$\frac{dy}{dx} = \frac{7}{4x^{3/4}} + \frac{1}{x}$$

(7) $y = x^2 \log x - e^x a^x$

Solution:

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 \log x - e^x a^x)$$

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 \log x) - \frac{d}{dx}(e^x a^x)$$

$$\frac{dy}{dx} = [x^2 \frac{d}{dx}(\log x) + (\log x)\frac{d}{dx}(x^2)] - [e^x \frac{d}{dx}(a^x) + a^x \frac{d}{dx}(e^x)]$$

$$\frac{dy}{dx} = x^2\left(\frac{1}{x}\right) + (\log x)(2x) - (e^x(a^x \log a) + a^x e^x)$$

$$\frac{dy}{dx} = x + (\log x)(2x) - e^x(a^x \log a) - a^x e^x$$

$$(\log x)] + (2e^x - 2\log x)[4\frac{d}{dx}(x^4) - 3\frac{d}{dx}(x^2) + 9\frac{d}{dx}(x) + \frac{d}{dx}(8)]$$

$$\frac{dy}{dx} = (4x^4 - 3x^2 + 9x + 8)(2e^x - \frac{2}{x}) + (2e^x - 2\log x)(16x^3 - 6x + 9)$$

(8) $y = (5 \log x - 2e^x)(4^x - 8e^x)$

Solution:

Diff. w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx}[5 \log x - 2e^x)(4^x - 8e^x)]$$

$$\frac{dy}{dx} = (5 \log x - 2e^x)\frac{d}{dx}(4^x - 8e^x) + (4^x - 8e^x)\frac{d}{dx}(5 \log x - 2e^x)$$

$$\frac{dy}{dx} = (5 \log x - 2e^x)[\frac{d}{dx}(4^x) - 8\frac{d}{dx}(e^x)] + (4^x - 8e^x)[5\frac{d}{dx}(\log x) - 2\frac{d}{dx}(e^x)]$$

$$\frac{dy}{dx} = (5 \log x - 2e^x)(4^x \log 4 - 8e^x) + (4^x - 8e^x)(5\left(\frac{1}{x}\right) - 2e^x$$

(9) $y = a^x . x^a + e^x . \log x$

Solution:

Diff. w. r. t. x

$$\frac{dy}{dx} = \frac{d}{dx}(a^x . x^a + e^x . \log x)$$

$$\frac{dy}{dx} = \frac{d}{dx}(a^x . x^a) + \frac{d}{dx}(e^x . \log x)$$

$$\frac{dy}{dx} = \left[a^x \frac{d}{dx}(x^a) + (x^a)\frac{d}{dx}(a^x)\right] + \left[e^x \frac{d}{dx}(\log x) + (\log x)\frac{d}{dx}(e^x)\right]$$

$$\frac{dy}{dx} = [a^x (a)x^{a-1} + (x^a)(a^x \log a)] + [e^x\left(\frac{1}{x}\right) + (\log x)e^x)$$

(10) $y = \dfrac{x - \sqrt{x}}{\sqrt{x} - 7}$

Solution:

Diff. w. r. t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x - \sqrt{x}}{\sqrt{x} - 7}\right)$$

Using,

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}u - u\frac{d}{dx}v}{v^2}$$

$$\frac{dy}{dx} = \frac{(\sqrt{x} - 7)\frac{d}{dx}(x - \sqrt{x}) - (x - \sqrt{x})\frac{d}{dx}(\sqrt{x} - 7)}{(\sqrt{x} - 7)^2}$$

$$\frac{dy}{dx} = \frac{(\sqrt{x} - 7)\left[\frac{d}{dx}(x) - \frac{d}{dx}(\sqrt{x})\right] - (x - \sqrt{x})\left[\frac{d}{dx}(\sqrt{x}) - \frac{d}{dx}(7)\right]}{(\sqrt{x} - 7)^2}$$

$$\frac{dy}{dx} = \frac{(\sqrt{x} - 7)\left(1 - \left(\frac{1}{2}\right)x^{-1/2}\right) - (x - \sqrt{x})\left[\left(\frac{1}{2}\right)x^{-1/2} - 0\right]}{(\sqrt{x} - 7)^2}$$

$$\frac{dy}{dx} = \frac{(\sqrt{x} - 7)\left(1 - \frac{1}{2x^{1/2}}\right) - (x - \sqrt{x})\left(\frac{1}{2x^{1/2}}\right)}{(\sqrt{x} - 7)^2}$$

(11) $y = \dfrac{x^3 - 2x^2 + 1}{x^3 - 3}$

Solution:

Diff. w. r. t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^3 - 2x^2 + 1}{x^3 - 3}\right)$$

$$\frac{dy}{dx} = \frac{(x^3 - 3)\frac{d}{dx}(x^3 - 2x^2 + 1) - (x^3 - 2x^2 + 1)\frac{d}{dx}(x^3 - 3)}{(x^3 - 3)^2}$$

$$\frac{dy}{dx} = \frac{(x^3 - 3)\left[\frac{d}{dx}(x^3) - 2\frac{d}{dx}(x^2) + \frac{d}{dx}(1)\right] - (x^3 - 2x^2 + 1)\left[\frac{d}{dx}(x^3) - \frac{d}{dx}(3)\right]}{(x^3 - 3)^2}$$

$$\frac{dy}{dx} = \frac{(x^3 - 3)\ (3x^2 - 4x + 0) - (x^3 - 2x^2 + 1)\ (3x^2 - 0)}{(x^3 - 3)^2}$$

(12) $y = \frac{(3x^2 + 2x + 7)}{5 - 4x}$

Solution:

Diff. w. r. t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{3x^2 + 2x + 7}{5 - 4x}\right)$$

$$\frac{dy}{dx} = \frac{(5 - 4x)\frac{d}{dx}(3x^2 + 2x + 7) - (3x^2 + 2x + 7)\frac{d}{dx}(5 - 4x)}{(5 - 4x)^2}$$

$$\frac{dy}{dx} = \frac{(5 - 4x)\left[3\frac{d}{dx}(x^2) + 2\frac{d}{dx}(x) + \frac{d}{dx}(7)\right] - (3x^2 + 2x + 7)\left[\frac{d}{dx}(5) - 4\frac{d}{dx}(x)\right]}{(5 - 4x)^2}$$

$$\frac{dy}{dx} = \frac{(5 - 4x)\ (6x + 2 + 0) - (3x^2 + 2x + 7)\ (0 - 4)}{(5 - 4x)^2}$$

(13) $y = (x - 5)(\log(15x^2 - e^x))$

Solution:

Diff. w. r. t. x

$$\frac{dy}{dx} = \frac{d}{dx}[(x - 5)\ (\log(15x^2 - e^x))]$$

$$\frac{dy}{dx} = (x - 5)\frac{d}{dx}(\log(15x^2 - e^x)) + (\log(15x^2 - e^x))\frac{d}{dx}(x - 5)$$

$$\frac{dy}{dx} = (x - 5)\left[\frac{1}{15x^2 - e^x}\right]\frac{d}{dx}(15x^2 - e^x) + (\log(15x^2 - e^x))\ (1)$$

$$\frac{dy}{dx} = \frac{(x - 5)}{15x^2 - e^x}(30x - e^x) + \log(15x^2 - e^x)$$

(14) $y = \left(a^{3x^2 - 7x + 8}\right)(\log(3x^2 - 7x + 8))$

Solution:

Diff. w. r. t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left(a^{3x^2 - 7x + 8}\right)(\log(3x^2 - 7x + 8))$$

$$\frac{dy}{dx} = \left(a^{3x^2 - 7x + 8}\right) \frac{d}{dx} (\log(3x^2 - 7x + 8)) + (\log(3x - 7x + 8)) \frac{d}{dx}$$

$$a^{3x^2 - 7x + 8}$$

$$\frac{dy}{dx} = \left(a^{3x^2 - 7x + 8}\right) \left(\frac{1}{3x^2 - 7x + 8}\right) \frac{d}{dx} (3x^2 - 7x + 8) + (\log(3x^2 - 7x + 8))$$

$$\left[a^{3x^2 - 7x + 8} \log a\right] \frac{d}{dx} (3x^2 - 7x + 8)$$

$$\frac{dy}{dx} = \frac{\left(a^{3x^2 - 7x + 8}\right)}{3x^2 - 7x + 8} (6x - 7) + (\log(3x^2 - 7x + 8)) \left(a^{3x^2 - 7x + 8} \log a\right) (6x - 7)$$

Application of Derivatives – I

(Increasing, Decreasing Functions)

(1) Find x for which the function f (x) = $2x^3 - 9x^2 + 12x + 5$ is (i) Increasing and (ii) Decreasing.

Solution:

$$f : \mathbb{R} \rightarrow \mathbb{R}$$

$$f(x) = 2x^3 - 9x^2 + 12x + 5$$

Diff. w. r. t. x

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}(2x^3 - 9x^2 + 12x + 5)$$

$$f'(x) = 6x^2 - 18x + 12$$

(I) For f(x) to be increasing $f'(x) > 0$.

$$6x^2 - 18x + 12 > 0$$

Dividing throughout by 6:

$$x^2 - 3x + 2 > 0$$

$$x^2 - 2x - x + 2 > 0$$

$$x(x - 2) - 1(x - 2) > 0$$

$$(x - 2)(x - 1) > 0$$

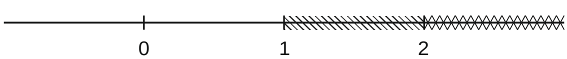
This gives the following two cases:

Case (i) $x - 2 > 0$ AND $x - 1 > 0$

OR

Case (ii) $x - 2 < 0$ AND $x - 1 < 0$

Case (i) gives $x > 2$ AND $x > 1$

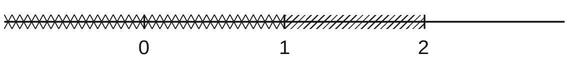


$$x \in (2, \infty) \text{ AND } x \in (1, \infty)$$

$$x \in (2, \infty) \cap (1, \infty)$$

$$x \in (2, \infty)$$

Case (ii) gives, $x < 2$ AND $x < 1$



(II) For f(x) to be decreasing $f'(x) < 0$.

$$6x^2 - 18x + 12 < 0$$

Dividing throughout by 6:

$$x^2 - 3x + 12 < 0$$

$$x^2 - 2x - x + 2 < 0$$

$$x(x - 2) - 1(x - 2) > 0$$

$$(x - 2)(x - 1) > 0$$

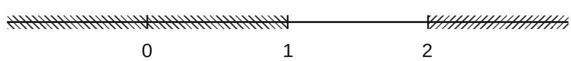
This gives the following two cases:

Case (i) $x - 2 > 0$ AND $x - 1 < 0$

OR

Case (ii) $x - 2 < 0$ AND $x - 1 > 0$

Case (i) gives $x > 2$ AND $x < 1$

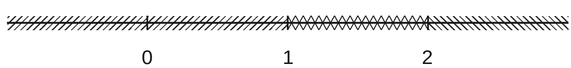


$$x \in (2, \infty) \text{ AND } x \in (-\infty, 1)$$

$$x \in (2, \infty) \cap (-\infty, 1)$$

$$x \in \{\phi\}$$

Case (ii) gives, $x < 2$ AND $x > 1$



$x \in (-\infty, 2)$ AND $x \in (-\infty, 1)$

$x \in (-\infty, 2) \cap (-\infty, 1)$

$x \in (-\infty, 1)$

Thus, form the two cases we conclude f(x) is increasing for all values of x:

$x \in (2, \infty) \cup (-\infty, 1)$

$x \in (-\infty, 1) \cup (2, \infty)$

$x \in (-\infty, 2)$ AND $x \in (1, \infty)$

$x \in (-\infty, 2) \cap (1, \infty)$

$x \in (1, 2)$

Thus, form the two cases we conclude f(x) is decreasing for all values of x:

$x \in \{\phi\} \cup (1, 2)$

$x \in (1, 2)$

(2) Total cost function is $C = Q^3 - 400Q^2 + 1000Q$ where Q = number of items produced. Find Q for which the average cost is decreasing.

Solution:

$$C : \mathbb{R}^+ \to \mathbb{R}$$

$$C(Q) = Q^3 - 400Q^2 + 1000Q$$

$$AC(Q) = \frac{C(Q)}{Q} = \frac{Q^3 - 400Q^2 + 1000Q}{Q} = Q^2 - 400Q + 1000$$

$$\therefore AC(Q) = \frac{Q^2 - 400Q + 1000}{Q}$$

Diff. w.r.t.

$$\therefore \frac{d}{dQ}(AC(Q)) = \frac{d}{dQ}(Q^2 - 400Q + 1000)$$

$$AC(Q) = 2Q - 400$$

For AC(Q) to be decreasing,

$$AC'(Q) < 0$$

$$2Q < 400$$

$$Q < 200$$

$$\Rightarrow x \in (0, 200)$$

(3) A manufacturing company produces x items at a total cost of Rs. (50 + 4x). The demand function is P = 30 – x where P is price and x is the number of units demanded. Find x for which total revenue is increasing. Also find x for which total profit is increasing.

Solution:

$$C : \mathbb{R}^+ \to \mathbb{R} \text{ and } R : \mathbb{R}^+ \to \mathbb{R}$$

$$C(x) = 50 + 4x$$

$$R(x) = PD$$

$$R(x) = (30 - x)x$$

$$R(x) = 30x - x^2$$

Diff w.r.t. x

$$\frac{d}{dx}(R(x)) = \frac{d}{dx}(30x - x^2)$$

$$R'(x) = 30 - 2x$$

For R(x) to be increasing

$$R'(x) > 0$$

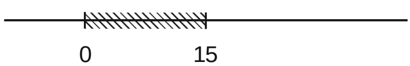
$$30 - 2x > 0$$

$$30 > 2x$$

$$\frac{30}{2} > x$$

$$\therefore x < 15$$

$$\Rightarrow x \in (0,\ 15)$$



$$\pi(x) = R(x) - C(x)$$

$$\pi(x) = (30x - x^2) - (50 + 4x)$$

$$\pi(x) = 30x - x^2 - 50 - 4x$$

$$\pi(x) = 26x - x^2 - 50$$

Dif. w.r.t. x

$$\frac{d}{dx}(\pi(x)) = \frac{d}{dx}(26x - x^2 - 50)$$

$$\pi'(x) = 26 - 2x$$

For $\pi(x)$ to be increasing

$$\pi'(x) > 0$$

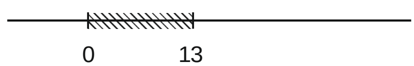
$$26 - 2x > 0$$

$$26 > 2x$$

$$\frac{26}{2} > x$$

$$\therefore x < 13$$

$$\Rightarrow x \in (0,\ 13)$$



(4) The total cost function is $C = 100 - 50x + x^2$ where x = the number of units produced. The demand function is $p = 200 - 4x$ where P is price and x is demand. Find x for which total cost is decreasing. Also find x for which total revenue is increasing.

Solution:

$$C : \mathbb{R}^+ \to \mathbb{R} \quad \text{and } R : \mathbb{R}^+ \to \mathbb{R}$$

$$C(x) = 100 - 50x + x^2$$

Diff. w. r. t. x

$$\frac{d}{dx}(C(x)) = \frac{d}{dx}(100 - 50x + x^2)$$

$$C'(x) = -50 + 2x$$

For C(x) to be decreasing

$$C'(x) < 0$$

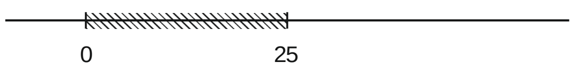
$$-50 + 2x < 0$$

$$2x < 50$$

$$x < \frac{50}{2}$$

$$x < 25$$

$$\Rightarrow x \in (0, 25)$$



$$R(x) = PD$$

$$R(x) = (200 - 4x)x$$

$$R(x) = 200x - 4x^2$$

Diff. w.r.t. x

$$\frac{d}{dx}(R(x)) = \frac{d}{dx}(200x - 4x^2)$$

$$R'(x) = 200 - 8x$$

For R(x) to be increasing

$$R'(x) > 0$$

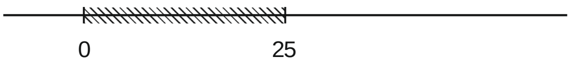
$$200 - 8x > 0$$

$$200 > 8x$$

$$x < \frac{200}{8}$$

$$x < 25$$

$$\Rightarrow x \in (0, 25)$$



(5) Total cost function $C(x) = x^3 - 248x^2 + 1500x$ where x = number of items produced. Find x for which the average cost is decreasing.

Solution:

$$C : \mathbb{R}^+ \to \mathbb{R}$$

$$C(x) = x^3 - 284x^2 + 1500x$$

$$AC(x) = \frac{C(x)}{x} = \frac{x^3 - 284x^2 + 1500x}{x}$$

$$= x^2 - 284x + 1500$$

$$AC(x) = x^2 - 284x + 1500$$

Diff. w. r. t. x

$$\frac{d}{dx}(AC(x)) = \frac{d}{dx}(x^2 - 284x + 1500)$$

$$AC'(x) = 2x - 284$$

For AC(x) to be decreasing

$$AC'(x) < 0$$

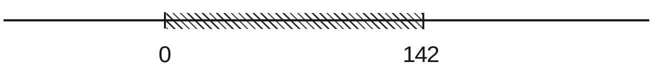
$$2x - 284 < 0$$

$$2x < 284$$

$$x < \frac{284}{2}$$

$$x < 142$$

$$\Rightarrow x \in (0, 142)$$



(6) The total cost function is $C(x) = 225 - 200x + 2x^2$ where x = the number of units produced. The demand function is $p = 700 - 7x$ where p is price and x is demand. Find x for which total cost is decreasing. Also find x for which total revenue is increasing.

Solution:

$$C : \mathbb{R}^+ \to \mathbb{R} \quad \text{and } R : \mathbb{R}^+ \to \mathbb{R}$$

$$C(x) = 225 - 200x + 2x^2$$

Diff. w. r. t. x

$$\frac{d}{dx}(C(x)) = \frac{d}{dx}(225 - 200x + 2x^2)$$

$$C'(x) = -200 + 4x$$

For C(x) to be decreasing

$$C'(x) < 0$$

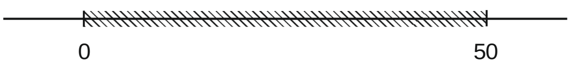
$$-200 + 4x < 0$$

$$4x < 200$$

$$x < \frac{200}{4}$$

$$x < 50$$

$$\Rightarrow x \in (0, 50)$$



$$R(x) = PD$$

$$R(x) = (700 - 7x)x$$

$$R(x) = 700x - 7x^2$$

Diff. w.r.t. x

$$\frac{d}{dx}(R(x)) = \frac{d}{dx}(700x - 7x^2)$$

$$R'(x) = 700 - 14x$$

For R(x) to be increasing

$$R'(x) > 0$$

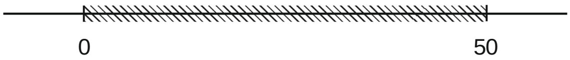
$$700 - 14x > 0$$

$$700 > 14x$$

$$x < \frac{700}{14}$$

$$x < 50$$

$$\Rightarrow x \in (0, 50)$$



Application of Derivatives – II

(Average and Marginal Concept)

(1) The cost function for a commodity is given by $C(Q) = 35 + 5Q - 2Q^2 + 2Q^3$. Find the marginal cost, the average cost and marginal average cost when $Q = 3$ units.

Solution:

$$C(Q) = 35 + 5Q - 2Q^2 + 2Q^3$$

$$AC(Q) = \frac{C(Q)}{Q} = \frac{35 + 5Q - 2Q^2 + 2Q^3}{Q}$$

$$AC(Q) = \frac{35}{Q} + 5 - 2Q + 2Q^2$$

Average cost when Q = 3.

$$AC(3) = \frac{35}{3} + 5 - 2(3) + 2(3)^2$$

$$= 11.667 + 5 - 6 + 2(9)$$

$$= 11.667 + 5 - 6 + 18$$

$$AC(3) = 28.667$$

$$MC(Q) = \frac{d}{dQ} C(Q)$$

$$MC(Q) = \frac{d}{dQ} (35 + 5Q - 2Q^2 + 2Q^3)$$

$$MC(Q) = 5 - 4Q + 6Q^2$$

Marginal cost when Q = 3.

$$MC(3) = 5 - 4(3) + 6(3)^2$$

$$= 5 - 12 + 6(9)$$

$$= 5 - 12 + 54$$

$$MC(3) = 47$$

$$\text{Marginal Average Cost} = \frac{d}{dQ} AC(Q)$$

$$= \frac{d}{dQ} \left(\frac{35}{Q} + 5 - 2Q + 2Q^2\right)$$

$$= -\frac{35}{Q^2} - 2 + 4Q$$

Marginal Average cost when Q = 3.

$$= -\frac{35}{(3)^2} - 2 + 4(3)$$

$$= -\frac{35}{9} - 2 + 12 = -3.889 - 2 + 12 = 6.111$$

(2) Find the marginal cost and marginal average cost functions for which each of the average cost function is as below:

(a) $AC(Q) = 1.5Q + 4 + \frac{46}{Q}$

(b) $AC(Q) = \frac{160}{Q} + 5 - 3Q + 2Q^2$

Solution:

(a) $AC(Q) = 1.5Q + 4 + \frac{46}{Q}$

$$\text{Marginal Average Cost} = \frac{d}{dQ}(AC(Q))$$

$$= \frac{d}{dQ}(1.5Q + 4 + \frac{46}{Q})$$

$$\text{Marginal Average cost} = 1.5 - \frac{46}{Q^2}$$

$$C(Q) = AC(Q) \times Q$$

$$= (1.5Q + 4 + \frac{46}{Q}) \times Q$$

$$C(Q) = 1.5Q^2 + 4Q + 46$$

$$MC(Q) = \frac{d}{dQ}(C(Q)) = \frac{d}{dQ}(1.5Q^2 + 4Q + 46)$$

$$MC(Q) = 3Q + 4$$

(b) $AC(Q) = \frac{160}{Q} + 5 - 3Q + 2Q^2$

$$\text{Marginal Average Cost} = \frac{d}{dQ}(AC(Q))$$

$$= \frac{d}{dQ}\left(\frac{160}{Q} + 5 - 3Q + 2Q^2\right)$$

$$\text{Marginal Average cost} = -\frac{160}{Q^2} - 3 + 4Q$$

$$C(Q) = AC(Q) \times Q$$

$$= \left(\frac{160}{Q} + 5 - 3Q + 2Q^2\right) \times Q$$

$$C(Q) = 160 + 5Q - 3Q^2 + 2Q^3$$

$$MC(Q) = \frac{d}{dQ}(C(Q)) = \frac{d}{dQ}(160 + 5Q - 3Q^2 + 2Q^3)$$

$$MC(Q) = 5 - 6Q + 6Q^2$$

(3) Output of x tons of a product has total cost function $C = x^3 - 4x^2 + 7x$. Find the output at which the average cost function is minimum. Also verify that at this output, AC = MC.

Solution:

$$C(x) = x^3 - 4x^2 + 7x$$

$$AC(x) = \frac{C(x)}{x} = \frac{x^3 - 4x^2 + 7x}{x} = x^2 - 4x + 7$$

For average cost to be minimum.

$$\frac{d}{dx}(AC(x)) = 0$$

$$\frac{d}{dx}(x^2 - 4x + 7) = 0$$

$$2x - 4 = 0$$

$$2x = 4$$

$$\therefore \quad x = 2$$

Now $\quad AC'(x) = 2x - 4$

Diff. w.r.t. x

$$\frac{d}{dx}(AC'(x) = \frac{d}{dx}(2x - 4)$$

$$AC''(x) = 2 > 0$$

For $\quad AC''(2) = (2)$

$$= 2 > 0$$

$\therefore$ Average cost is minimum at output 2.

$$\therefore \quad AC(x) = x^2 - 4x + 7$$

$$AC(2) = (2)^2 - 4(2) + 7$$

$$= 4 - 8 + 7$$

$$AC(2) = 3$$

$$MC(x) = \frac{d}{dx}(C(X)) = \frac{d}{dx}(x^3 - 4x^2 + 7x)$$

$$MC(x) = 3x^2 - 8x + 7$$

For output = 2.

$$MC(2) = 3(2)^2 - 8(2) + 7$$

$$= 3\,(4) - 8(2) + 7$$

$$MC(2) = 12 - 16 + 7 = 3$$

$\therefore$ This shows that AC = MC when the output is 2.

(4) The total revenue of a firm is given by $R(x) = 15x - 2x^2 - x^3$ where x is the quantity demanded. Find the average revenue and the marginal revenue when demand is x = 2.

Solution:

$$R(x) = 15x - 2x^2 + x^3$$

$$AR(x) = \frac{R(x)}{x} = \frac{15x - 2x^2 + x^3}{x} = 15 - 2x + x^2$$

$$AR(x) = 15 - 2x - x^2$$

Put x = 2.

$$AR(2) = 15 - 2(2) - (2)2$$

$$= 15 - 4 - 4$$

$$AR(2) = 7$$

$$MR(x) = \frac{d}{dx}(R(x))$$

$$= \frac{d}{dx}(15x - 2x^2 + x^3)$$

$$MR(x) = 15 - 4x + 3x^2$$

Put x = 2.

$$MR(2) = 15 - 4(2) - 3(2)^2$$

$$= 15 - 8 - 3(4)$$

$$= 15 - 8 - 12$$

$$MR(2) = -5$$

Application of Derivatives – III

(Maxima and Minima)

(1) Divide 100 into two parts such that the sum of square of one and twice the other is a minimum.

Solution:

Let x and (100 – x) be the two parts by dividing 100.

According to the condition,

$$f(x) = x^2 + 2(100 - x)$$

$$f(x) = x^2 + 200 - 2x$$

Diff w.r.t x

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}(x^2 + 200 - 2x)$$

$$f'(x) = 2x - 2$$

For f(x) to be minimum.

$$f'(x) = 0$$

$$\Rightarrow 2x - 2 = 0$$

$$2x = 2$$

$$\boxed{x = 1}$$

$$\Rightarrow 100 - x = 100 - 1 = 99$$

$$f'(x) = 2x - 2$$

Diff. w.r.t. x

$$\frac{d}{dx}(f'(x)) = \frac{d}{dx}(2x - 2)$$

$$f''(x) = 2 > 0$$

Now,

$$f''(1) = (2)$$

$$= 2 > 0$$

$\therefore$ f(x) is minimum at x = 1.

$\therefore$ The required two parts are 1, 99.

(2) Prove that, of all rectangles having the same area, the square has the minimum perimeter.

Solution:

$$P = 2\,(l + b)$$
$$A = lb$$
$$b = \frac{A}{l}$$
$$P = 2\left(l + \frac{A}{l}\right)$$
$$P = 2l + \frac{2A}{l}$$

Diff w.r.t. l

$$\frac{dP}{dl} = \frac{d}{dl}\left(2l + \frac{2A}{l}\right)$$
$$\frac{dP}{dl} = 2 - \frac{2A}{l^2}$$
$$\frac{dP}{dl} = 2\left(1 - \frac{A}{l^2}\right)$$

For P to be minimum,

$$\frac{dP}{dl} = 0$$
$$2\left(1 - \frac{A}{l^2}\right) = 0$$
$$1 - \frac{A}{l^2} = 0$$
$$1 = \frac{A}{l^2}$$
$$\therefore A = l^2$$
$$\Rightarrow lb = l^2$$
$$\Rightarrow b = l$$

$\Rightarrow$ Square has the minimum perimeter.

(3) The demand function is given by $P = 108 + 9D - 2D^2$ where P is the price of commodity and D is its demand. Find the maximum value of revenue function.

Solution:

$$R(D) = PD$$
$$R(D) = (108 + 9D - 2D^2)D$$
$$R(D) = 108D + 9D^2 - 2D^3$$

Diff. w.r.t. D.

$$\frac{d}{dD}(R(D)) = \frac{d}{dD}(108D + 9D^2 - 2D^3)$$
$$R'(D) = 108 + 18D - 6D^2$$

For R(D) to be maximum.

$$R'(D) = 0$$
$$108 + 18D - 6D^2 = 0$$

Dividing throughout by 6,

$$18 + 3D - D^2 = 0$$
$$18 - 3D + 6D - D^2 = 0$$
$$3(6 - D)(3 + D) = 0$$

$\Rightarrow \boxed{D = 6}$ or $\boxed{D = -3}$ are the critical points for the R(D) function.

Discard $D = -3$, since quantity cannot be negative.

$$R'(D) = 108 + 18D - 6D^2$$

Diff. w.r.t. x

$$\frac{d}{dx}(R'(D)) = \frac{d}{dx}(108 + 18D - 6D^2)$$
$$R''(D) = 18 - 12D$$

Now, $$R''(D) = (18 - 12D)$$
$$= 18 - 12(6)$$
$$= 18 - 72$$
$$= -54 < 0$$

Thus, the function is maximum at D = 6 and the maximum value is given by,

$$R(D) = 108D + 9D^2 - 2D^3$$
$$R(6) = 108(6) + 9(6)^2 - 2(6)^3$$
$$= 108(6) + 9(36) - 2(216)$$
$$= 648 + 324 - 432 = 540$$

Hence, the function is maximum at D = 6 and the maximum value is 540.

(4) A manufacturer finds that the total cost of producing x units daily is Rs. $(260x - 3x^2)$ and the price is Rs. $(500 - 20x)$ per unit. Find the output x for which the profit is maximum.

Solution:

$$C(x) = 260x - 3x^2$$
$$R(x) = PD$$
$$R(x) = (500 - 20x)x$$
$$R(x) = 500x - 20x^2$$

Let $\pi(x)$ be the profit function.

$$\pi(x) = R(x) - C(x)$$
$$= 500x - 20x^2 - (260x - 3x^2)$$
$$= 500x - 20x^2 - 260x + 3x^2$$
$$\pi(x) = 240x - 17x^2$$

Diff. w.r.t. x

$$\frac{d}{dx}(\pi(x)) = \frac{d}{dx}(240x - 17x^2)$$
$$\pi'(x) = 240 - 34x$$

For $\pi(x)$ to be maximum.

$$\pi'(x) = 0$$
$$240 - 34x = 0$$
$$34x = 240$$

$\boxed{x = \frac{120}{17}}$ is the critical point for the function.

$$\pi'(x) = 240 - 34x$$

Diff. w.r.t x

$$\frac{d}{dx}(\pi'(x)) = \frac{d}{dx}(240 - 34x)$$
$$\pi''(x) = -34 < 0$$

So, $$\pi''\left(\frac{120}{17}\right) = -34 < 0$$

Thus, the function is maximum at $x = \frac{120}{17}$ and the maximum value is given by,

$$\pi(x) = 240x - 17x2$$
$$\pi\left(\frac{120}{17}\right) = 240\left(\frac{120}{17}\right) - 17\left(\frac{120}{17}\right)^2$$
$$= \frac{28800}{17} - \frac{14400}{17}$$
$$= \frac{28800 - 14400}{17} = \frac{14400}{17} = 847.06$$

Hence, the function is maximum at x = $\frac{120}{17}$ and the maximum value is 847.06.

(5) Output of x tons of a product has total cost function $C = 3x^3 - 36x^2 + 108x$. Find the output at which the average cost function is minimum. Also verify that at this output AC = MC.

Solution:

$$AC(x) = \frac{C(x)}{x} = \frac{3x^3 - 36x^2 + 108x}{x}$$

$$= 3x^2 - 36x + 108$$

Diff. w. r. t. x

$$\frac{d}{dx}(AC(x)) = \frac{d}{dx}(3x^2 - 36x + 108)$$

$$AC'(x) = 6x - 36$$

For AC(x) to be minimum,

$$AC'(x) = 0$$

$$6x - 36 = 0$$

$$6x = 36$$

$\boxed{x = 6}$ is the critical point for AC(x):

$$AC'(x) = 6x - 36$$

Diff w.r.t. x

$$\frac{d}{dx}(AC'(x)) = \frac{d}{dx}(6x - 36)$$

$$AC''(x) = 6$$

So, $$AC''(6) = (6)$$

$$= 6 > 0$$

Thus, the function is minimum at x = 6 and the minimum value is given by,

$$AC(x) = 3x^2 - 36x + 108$$

$$AC(6) = 3(6)^2 - 36(6) + 108$$

$$= 108 - 216 + 108$$

$$AC(6) = 0 \quad \text{.........} \ \textbf{(1)}$$

$$MC(x) = \frac{d}{dx}(C(x))$$

$$= \frac{d}{dx}(3x^3 - 36x^2 + 108x)$$

$$MC(x) = 9x^2 - 72x + 108$$

$$MC(6) = 9(6)^2 - 72(6) + 108$$

$$= 324 - 432 + 108$$

$$MC(6) = 0 \quad \text{.........} \ \textbf{(2)}$$

AC = MC at this output.

(6) The total cost function is given by $C = x^3 - 9x^2 + 24x + 17$. Find x for which the cost is minimum.

Solution:

$$C(x) = x^3 - 9x^2 + 24x + 17$$

Diff. w. r. t. x

$$\frac{d}{dx}(C(x)) = \frac{d}{dx}(x^3 - 9x^2 + 24x + 17)$$

$$C'(x) = 3x^2 - 18x + 24$$

for C(x) to be minimum.

$$C'(x) = 0$$

$$3x^2 - 18x + 24 = 0$$

Divide throughout by 3,

$$x^2 - 6x + 8 = 0$$

$$x(x-4) - 2(x-4) = 0$$

$$(x-4)(x-2) = 0$$

$\Rightarrow$ x = 4 or x = 2 are the critical points for C(x).

$$C'(x) = 3x^2 - 18x + 24$$

Diff. w.r.t x

$$\frac{d}{dx}(C'(x)) = \frac{d}{dx}(3x^2 - 18x + 24)$$

$$C''(x) = 6x - 18$$

Now, $C''(4) = (6x - 18)$

$$= 6(4) - 18$$

$$= 24 - 18$$

$$= 6 > 0$$

And

$$C''(2) = (6x - 18)$$

$$= 6(2) - 18$$

$$= 12 - 18$$

$$= -6 < 0$$

The function is minimum at x = 4 and the minimum value is given by,

$$C(x) = x^3 - 9x^2 + 24x + 17$$

$$C(4) = (4)^3 - 9(4)^2 + 24(4) + 17$$

$$= 64 - 144 + 96 + 17$$

$$C(4) = 33$$

Hence, the function is minimum at x = 4 and the minimum value is 33.

(7) The total cost function is $C = Q^3 - 4Q^2 + 15Q$, where Q is the number of units produced. Find the output at which AC is minimum and the value of minimum AC.

Solution:

$$AC(Q) = \frac{C(Q)}{Q} = \frac{Q^3 - 4Q^2 + 15Q}{Q} = Q^2 - 4Q + 15$$

$$AC(Q) = Q^2 - 4Q + 15$$

Diff. w.r.t. Q

$$\frac{d}{dQ}(AC(Q)) = \frac{d}{dQ}(Q^2 - 4Q + 15)$$

$$AC'(Q) = 2Q - 4$$

For AC(Q) to be minimum,

$$AC'(Q) = 0$$

$$2Q - 4 = 0$$

$$2Q = 4$$

$\boxed{Q = 2}$ is the critical point of AC(Q).

$$AC'(Q) = 2Q - 4$$

Diff. w.r.t. Q

$$\frac{d}{dQ}(AC'(Q)) = \frac{d}{dQ}(2Q - 4)$$

$$AC''(Q) = 2$$

So, $AC''(2) = (2)$

$$= 2 > 0$$

The function is minimum at Q = 2 and the minimum value is given by,

$$AC(Q) = Q^2 - 4Q + 15$$

$$AC(2) = (2)^2 - 4(2) + 15$$

$$= 4 - 8 + 15$$

$$AC(2) = 11$$

Hence, the function is minimum at Q = 2 and the minimum value is 11.

Application of Derivatives – IV

(Elasticity)

(1) The demand function for a commodity is given by $D = \dfrac{x + 2}{x - 1}$. Find the elasticity of demand when price is 3 units.

Solution:

$$D = \frac{x + 2}{x - 1}$$

Diff. w.r.t. x

$$\frac{dD}{dx} = \frac{d}{dx}\left(\frac{x + 2}{x - 1}\right)$$

$$\frac{dD}{dx} = \frac{(x - 1)\dfrac{d}{dx}(x + 2) - (x + 2)\dfrac{d}{dx}(x - 1)}{(x - 1)^2}$$

$$\frac{dD}{dx} = \frac{(x - 1)(1) - (x + 2)(1)}{(x - 1)^2}$$

$$\frac{dD}{dx} = \frac{x - 1 - x - 2}{(x - 1)^2}$$

$$\frac{dD}{dx} = \frac{-3}{(x - 1)^2}$$

$$\therefore \eta = \frac{-x}{D} \cdot \frac{dD}{dx}$$

$$\eta = \frac{-x}{\dfrac{x + 2}{x - 1}} \cdot \frac{-3}{(x - 1)^2}$$

Put x = 3

$$\eta = \frac{-3}{\dfrac{(3 + 2)}{(3 - 1)}} \cdot \frac{-3}{(3 - 1)^2}$$

$$= \frac{9}{5 \times 2}$$

$$= \frac{9}{10}$$

$$= 0.9$$

(2) If MR is Rs. 26 and elasticity of demand with respect to price is 3, find AR.

Solution:

MR = 26, η = 3

$$\therefore \text{MR} = \text{AR}\left(1 - \frac{1}{\eta}\right)$$

$$\therefore 26 = \text{AR}\left(1 - \frac{1}{3}\right)$$

$$\therefore 26 = \text{AR}\left(\frac{3-1}{3}\right)$$

$$\therefore 26 = \text{AR}\left(\frac{2}{3}\right)$$

$$\therefore \text{AR} = \frac{26 \times 3}{2}$$

$$\therefore \text{AR} = 39$$

(3) Find the elasticity of demand with respect to price at p = 2 if the demand function is given be $D = 100 - 2p - 3p^2$

Solution:

$$D = 100 - 2p - 3p^2$$

Diff. w.r.t. p

$$\frac{dD}{dp} = \frac{d}{dp}(100 - 2p - 3p^2)$$

$$\frac{dD}{dp} = 0 - 2 - 6p$$

$$\frac{dD}{dp} = -2 - 6p$$

$$\eta = \frac{-p}{D} \cdot \frac{dD}{dp}$$

$$\eta = \frac{-p}{100 - 2p - 3p^2} \cdot (-2 - 6p)$$

Put p = 2.

$$\eta = \frac{-2}{100 - 2(2) - 3(2)^2} \cdot (-2 - 6(2))$$

$$\eta = \frac{-2}{100 - 4 - 3(4)} \cdot (-2 - 12)$$

$$\eta = \frac{-2}{100 - 4 - 12} \cdot (-14)$$

$$\eta = \frac{28}{84}$$

$$\eta = 0.3333$$